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Holomorphic Differentials as Functions of Moduli

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The purpose of this note is to strengthen the results of [3] and to indicate a very brief derivation of some theorems announced without proof in [1] and [3].

We begin by indicating a correction to [3]. Let S_1 and S_2 be Riemann surfaces, f an orientation preserving (orientation reversing) homeomorphism of bounded excentricity of S_1 onto S_2 and $[f]$ the homotopy class of f; then $(S_1,[f],S_2)$ is called an even (odd) coupled pair of Riemann surfaces. The definition of equivalence of such pairs given in [3] is imprecise and garbled by misprints. The correct definition reads: $(S_1,[f],S_2)$ and $(S_1',[f'],S_2')$ are called <u>equivalent</u> if there exist conformal homeomorphisms h_1 and h_2 with $h_1(S_1) = S_1'$, $h_2(S_2) = S_2'$ and $[h_2 f] = [f' h_1]$; the two pairs are called <u>strongly equivalent</u> if $S_2' = S_2$ and there exists a conformal homeomorphism h with $h(S_1) = S_1'$ and $[f] = [h'f]$. If S_0 is a Riemann surface, then the Teichmüller space $T(S_0)$ can be thought of as the set of strong equivalence classes of even pairs $(S,[f],S_0)$ (and not of simple equivalence classes as stated in [3]).[1]

From now on we assume that S_0 is a fixed closed Riemann surface of genus $g > 1$, and we write T instead of $T(S_0)$. T has a natural complex analytic structure and can be represented as a bounded domain, homeomorphic to a ball, in complex number space with coordinates (moduli) $\tau_1, \ldots, \tau_{3g-3}$ (cf. [1,2]). Points of T will be denoted by τ. We may assume that S_0 is given as the unit disc modulo a fixed-point-free Fuchsian group, and that $\tau=0$ corresponds to the pair $(S_0,[\text{identity}], S_0)$.

The purpose of this note is to generalize the results of [1] and to indicate a very brief application of some theorems announced without proof in [2] and [3].

We begin by introducing a generalization of [1]. Let G_1 and G_2 be [illegible] topological groups. [illegible] [illegible] of G_1 onto G_2 and (ii) the [illegible] [illegible] is called an open [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] are called [illegible] [illegible] [illegible] [illegible] [illegible] are called strongly [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

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THEOREM I. *One can associate to every* $\tau \in T$ *a bounded Jordan domain* $D(\tau)$ *and* $2g$ *Möbius transformations* $z \to A_j(z,\tau)$, $z \to B_j(z,\tau)$, $j = 1, \ldots, g$, *such that the following conditions are satisfied.*

(i) *The boundary curve of* $D(\tau)$ *admits the parametric representation* $z = \sigma(e^{i\theta}, \tau)$, $0 \leq \theta \leq 2\pi$, σ *depending holomorphically on* τ. $D(0)$ *is the unit disc.*

(ii) *The* A_j *and* B_j *depend holomorphically on* τ *and satisfy the relation*

$$\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = 1 \tag{1}$$

For every fixed $\tau \in T$ *they generate, with the single defining relation (1), a fixed-point-free discrete group* $G(\tau)$ *of conformal self-mappings of* $D(\tau)$, *so that* $S(\tau) = D(\tau)/G(\tau)$ *is a closed Riemann surface of genus* g. $S(0)$ *is the surface* S_0.

(iii) Denote by $\alpha(\tau)$ the basis of the fundamental group of $S(\tau)$ defined by $A_1, \ldots, B_g$, and by f_τ a quasiconformal mapping of $S(\tau)$ onto $S(0)$ which takes $\alpha(\tau)$ into $\alpha(0)$. Then the point τ corresponds to the pair $(S(\tau), [f_\tau], S_0)$.

This statement differs from Theorem 2 in [3] primarily by the boundedness condition for $D(\tau)$ and can be obtained from that theorem without much difficulty.

We denote by M the domain in complex number space of $3g-2$ dimensions which consists of points (z,τ) with $z \in D(\tau)$ and $\tau \in T$. By Theorem 3 in [3] M is holomorphically equivalent to $T(S_0 - \{p\})$ for a fixed $p \in S_0$.

THEOREM 1. [illegible]

[illegible]

(ii) [illegible]

(1) [illegible]

[illegible]

(iii) [illegible]

[illegible]

[illegible]

We denote by $W_q(\tau)$ the (complex) vector space of holomorphic functions $\varphi(z)$, $z \in D(\tau)$, for which $\varphi(z)dz^q$ is invariant under $G(\tau)$; this is the same as the space of q-dimensional holomorphic differentials on $S(\tau)$, so that $\dim W_q(\tau) = 0, 1, g$, or $(2q-1)(g-1)$ according to whether $g \geq 0$, $g=0, g=1$, or $g>1$. In $W_1(\tau)$ there exist g distinguished elements, $p_k(z,\tau)$, determined by the conditions

$$(2) \qquad \int_z^{A_i(z,\tau)} p_k(z',\tau)dz' = \delta_{ik};$$

these correspond to the normalized Abelian differentials of the first kind on $S(\tau)$ belonging to the 'canonical' homology basis $a(\tau)$ determined by $\alpha(\tau)$. The period matrix of $S(\tau)$ belonging to $a(\tau)$ will be denoted by $Z(\tau)$. It has the elements

$$Z_{ik}(\tau) = \int_z^{B_i(z,\tau)} p_k(z',\tau)dz'$$

and is a point in the Siegel space of symmetric matrices with positive definite imaginary part.

We denote by $\underline{W}_q$ the vector space of holomorphic functions $\Phi(z,\tau)$, $(z,\tau) \in M$, which belong to $W_q(\tau)$ for every fixed $\tau \in T$.

THEOREM II. <u>Every element of $W_q(\tau)$ is a restriction of an element of $\underline{W}_q$.</u>

Proof. Assume that $q > 2$. Let C_j, $j=1, 2, \ldots$, be a complete system of non equivalent (with respect to (1)) words in the letters $A_1, \ldots, B_g$. If $P(t)$ is a polynomial, then the Poincaré series

$$(3) \qquad \sum_{j=1}^{\infty} P(C_j(z,\tau))(\partial C_j(z,\tau)/\partial z)^q$$

converges normally in M and its sum belongs to $\underline{W}_q$. On the other hand, since $D(\tau)$ is a bounded Jordan domain and $G(\tau)$ has a compact fundamental region, Theorem 4 in [4] implies that, for a fixed τ, every element of $W_q(\tau)$ is of the form (3).

For $q = 1$ we shall show that every p_j belongs to $\underline{W}_1$ (i.e. that the normalized Abelian differentials are holomorphic functions of the moduli).

THEOREM III. The functions $p_k(z,\tau)$, $k=1, \ldots, g$, are holomorphic in M.

Proof. It suffices to consider p_1. We shall show that in a neighborhood of a fixed but arbitrary point $\tau_0 \in T$ we have an identity of the form

$$(4) \qquad p_1(z,\tau) = \phi(z,\tau)^{-1} \sum_{j=1}^{5g-5} c_j(\tau)\, \phi_j(z,\tau)$$

where the c_j are holomorphic, $\phi \in \underline{W}_2$, and the ϕ_j are elements of $\underline{W}_3$. We first choose ϕ so that $\phi(z,\tau_0)$ vanishes at $4g-4$ points z_i which are not equivalent under $G(\tau_0)$. This is possible since the 'general' holomorphic quadratic differential on $S(\tau)$ has only simple zeros (Bertini) and hence exactly $4g-4$ of those. There exist $4g-4$ holomorphic functions $\zeta_i(\tau)$ defined near τ_0, such that $\zeta_i(\tau_0)=0$ and $\phi(z_i + \zeta_i(\tau),\tau) = 0$. In order that the right hand side of (4) belong to $W_1(\tau)$ it is necessary and sufficient that

$$\sum_{j=1}^{5g-5} c_j(\tau)\, \phi_j(z_i + \zeta_i(\tau),\tau) = 0, \quad i = 1, \ldots, 4g-4,$$

and one sees at once that any $4g-5$ of these equations imply the $(4g-4)$-th. In order that (4) hold near τ_0 the c_j must satisfy g

additional linear equations which are obtained from (1) by setting $k=1$ and choosing a fixed point z and fixed paths of integration, avoiding the points z_i. The resulting linear system, with holomorphic coefficients, for the unknown functions c_j, is uniquely solvable at τ_0 if the functions $\overset{\tau}{\theta}_1, \ldots, \overset{\tau}{\theta}_{15g-5}$ are chosen so as to be linearly independent for $\tau = \tau_0$. In this case the equations are uniquely solvable for τ close to τ_0, and the solutions depend holomorphically on τ.

We proceed to derive some consequences from Theorems II and III.

(a) <u>The functions</u>

$$f_{ij} = p_i/p_j, \; f_{ijk} = f_k^{-1} \partial \log f_{ij}/\partial z$$

<u>are meromorphic in</u> M. This proves Theorem J in [1]. It is classical that every meromorphic function in $D(\tau)$ which is automorphic under $G(\tau)$ can be expressed rationally in terms of the functions f_{ij}, f_{ijk} (and even in terms of the f_{ij} alone if $S(\tau)$ is not hyperelliptic). Thus we obtain a proof of Theorem 4 in [3] which asserts the existence of finitely many meromorphic functions of the moduli and of an additional complex variable which uniformize simultaneously all algebraic curves of genus $g > 1$.

(b) Let us choose $(2q-1)(g-1)$ elements of $\underline{W}_q, q>1$ (or g elements of $\underline{W}_1$) which are linearly independent for $\tau = \tau_0$, and let $w(z,\tau)$ denote their Wronskian with respect to z. For a fixed τ close to τ_0 the zeros of $w(z,\tau)$ are precisely the Weierstrass points of $S(\tau)$, in the classical sense if $q = 1$, in the sense of Petersson if $q > 1$ (cf. the definition in [4]). Since w is a holomorphic function in M

we conclude that the Weierstrass points on a closed Riemann surface depend holomorphically on the moduli (cf. Rauch [6], Röhrl [7]).

(c) Now let $w(z,\tau)$ denote the Wronskian of an arbitrary set of dim $W_q(\tau)$ elements of $\underline{W}_q$ and let N denote the set of those $\tau \in T$ for which $w(z,\tau) \equiv 0$. If z_o is not a Weierstrass point of $S(\tau_o)$, then there is a neighborhood of τ_o in which the points of N are precisely the zeros of $w(z_o,\tau)$. We conclude that N is either empty, or the whole domain T, or an analytic subvariety of T of codimension 1.

(d) Let H denote the set of those $\tau \varepsilon T$ for which $S(\tau)$ is hyperelliptic. For $\tau \varepsilon T - H$ every element of $W_q(\tau)$ can be written as a homogeneous polynomial in the p_j (M. Noether). For $\tau \varepsilon H$ the subspace of $W_q(\tau)$ consisting of homogeneous polynomials in elements of $W_1(\tau)$ has dimension $q(g-1)-1$. But H is an analytic subvariety of T of dimension $2g-1$, so that, noting (c), we obtain the following complement to Noether's theorem: for $g > 2$ and $q > 1$ there exist no fixed set of $(2q-1)(g-1)$ homogeneous polynomials of degree q in normalized Abelian differentials of the first kind which spans the space of holomorphic differentials of dimension q on all non-hyperelliptic closed Riemann surfaces of genus g.

(e) The mapping $\tau \rightarrow Z(\tau)$ of the Teichmüller space into the Siegel space is holomorphic. This follows at once from Theorem III, and also by using the coordinates in T defined in [1] in conjunction with Rauch's variational formulas [5]. These formulas also show

that the mapping of T into a (3g-3) -dimensional subspace of the Siegel space,

$$\tau \rightarrow \left\{ \sum_{i,k=1}^{3g-3} \gamma_{j,ik} Z_{ik}(\tau), \ j = 1, \ldots, 3g-3 \right\}$$

is one-to-one near a point τ_o if and only if the 3g-3 functions

$$\sum_{i,k=1}^{3g-3} \gamma_{j,ik} p_i(z,\tau_o) p_k(z,\tau_o)$$

are linearly independent. This shows that near every non-hyperelliptic surface a properly chosen set of 3g-3 periods Z_{ik} can serve as a set of local moduli (Rauch). On the other hand, (d) implies a complementto Rauch's theorem: <u>no fixed set of</u> 3g-3 <u>linear combinations of periods can serve as a set of moduli near every non-hyperelliptic closed Riemann surface of genus</u> g > 2.

FOOTNOTE

1) We also note the following errata to [2,3]. On p. 94, l.19, replace (ζ) by $\mu(\zeta)$. On p. 96, l.15, replace the subscript j by 2j. On p. 97, l.21, replace C_r by C^r. On p. 100, l.4, replace 'covering' by 'covering space'. On p. 103, equation (9) replace the exponent 3g-3n+n by 3g-3+n.

REFERENCES

1. Lipman Bers, Spaces of Riemann surfaces, Proc, Intern. Congr. of Math., Edinburgh, 1958, pp. 349-361.

2. - , Spaces of Riemann surfaces as bounded domains, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 98-103.

3. - , Simultaneous uniformization, ibid. pp.94-97.

4. - , Completeness theorems for Poincare series in one variable, to appear.

5. H. E. Rauch, On the transcendental moduli of algebraic Riemann surfaces, Proc. Nat. Acad. Sci. U.S.A., vol. 41 (1955) pp. 42-49.

6. - , Weierstrass points, branch points, and moduli of Riemann surfaces, Comm. Pure Appl. Math. vol. 13 (1959) pp. 543-560.

7. Helmut Röhrl, On holomorphic families of fiber bundles over the Riemannian sphere, to appear.

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